Stability of the dynamics of an asymmetric neural network

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Abstract

We study the stability of the dynamics of a network of n formal neurons interacting through an asymmetric matrix with independent random Gaussian elements of the type introduced by Rajan and Abbott ((([10]). The neurons are represented by the values of their electric potentials $x_i, i = 1, \ldots, n$. Using the approach developed in a previous paper by us ([7]) we obtain sufficient conditions for diverging synchronized behavior and for stability.

1 Introduction

The dynamic of a system of neurons described by their electric potentials $x_i(t), i = 1, \ldots, n$ interacting linearly through a random matrix has been extensively studied in the past literature and received increased attention in the last times, see for example ([1], [2], [5]), [3]). The first statement about the stability of the solution of the system

$$\mathbf{x}' = -\kappa \mathbf{x} + \mathbf{J}'\mathbf{x}, \tag{1.1}$$

was enunciated by May ([9]). Here $\mathbf{J}'$ is a real symmetric $n \times n$ matrix with independent gaussian elements and $E\mathbf{J}'_{ij} = 0$, $E\mathbf{J}'^2_{ij} = 1/n$. The conjecture was that if $\kappa > \lambda_{\max}$, where $\lambda_{\max}$ is the maximum eigenvalue of $\mathbf{J}'$, then the solutions of the system (1.1) were stable. This conjecture has been proved by us many years later in the paper ([7]) where the self-averaging property of the system have been used. In particular we introduced the random counting measure

$$\mathcal{N}_n(\lambda, t) = n^{-1} \sharp\{x_i(t) \leq \lambda\} = n^{-1} \sum_{i=1}^{n} \theta(\lambda - x_i(t)), \tag{1.2}$$

where $\theta(x)$ is the standard Heaviside function. This function $\mathcal{N}_n$ counts the fraction of the electric potentials $x_i(t)$ which are less than a given threshold λ. The self-averaging property of $\mathcal{N}_n$ means that $\mathcal{N}_n(t) \to E\mathcal{N}_n(t)$ in the L^2 norm with respect to the probability measure of the gaussian matrix, E being the expectation with respect to the probability of all the random entries of the matrix $\mathbf{J}'$. In ([7]) we were able to proof this property, so the random measure becomes asymptotically a gaussian distribution function with mean value $a(t)$ and dispersion $\sigma(t)$:

$$\lim_{n\to\infty} E\{\mathcal{N}_n(\lambda, t)\} = \int_{-\infty}^{\lambda} dx \frac{e^{-(x-a(t))^2/2\sigma(t)}}{\sqrt{2\pi\sigma(t)}}. \tag{1.3}$$

The results of the calculations were that $a(t) = e^{-\kappa t}$ and $\sigma(t) = e^{-2\kappa t} J_0(wt)$ where J_0 is the Bessel function of zero order. From the asymptotic behavior of J_0 we get that if $\kappa > w$ $\sigma(t)$ goes to zero and we get a stable solution. It is a quite remarkable coincidence that this nice result

depends on the self-averaging property of the system, this shows the real power of such property if one reminds all the rigorous properties which have been possible to show in the field of Statistical Mechanics of disordered systems. In this paper we look for analogous results when a different matrix $\mathbf{J}'$ is considered. The elements of $\mathbf{J}'$ are still independent but each row of the matrix have mean values depending on the column index:

$$E\{J'_{ij}\} = a \cdot \left\{ \begin{array}{ll} \mu_I/n^{1/2}, & j = 1, \dots [fn], \\ \mu_E/n^{1/2}, & j = [fn]+1, \dots, n. \end{array} \right\} \tag{1.4}$$

The first $[fn]$ columns represent inhibitory interaction ($\mu_I < 0$) while the other $n - [fn]$ are excitatory interaction ($\mu_E > 0$), thus each neuron i receives $[fn]$ inhibitory inputs of the same type from the other neurons and $n - [fn]$ excitatory inputs form the other neurons, the excitation and the inhibition do not depending on the particular neuron i. With this choice the matrix $\mathbf{J}'$ is asymmetric and the variances of the matrix elements of $\mathbf{J}'$ also follow the same choice:

$$E\{(J'_{ij} - E\{J'_{ij}\})^2\} = n^{-1}\sigma_j = \left\{ \begin{array}{ll} \sigma_I/n, & j = 1, \dots [fn], \\ \sigma_E/n, & j = [fn]+1, \dots, n. \end{array} \right. \tag{1.5}$$

Thus we look at the same property as before in the case of this new matrix which includes inhibitory and excitatory inputs which is nearer to realistic neural interactions. In order to understand better the new kind of stability properties that we obtain let us introduce some more definitions. Let $\mathbf{m} = (m_1, \dots, m_n)$ be the vector defined by

$$m_i = \left\{ \begin{array}{ll} \mu_I/n^{1/2}, & j = 1, \dots [fn] \\ \mu_E/n^{1/2} & , j = [fn]+1, \dots, n \end{array} \right. \tag{1.6}$$

and $\mathbf{M}$ be the matrix with all rows equal to $\mathbf{m}$. Then in the paper, following the ideas of (([10]), we introduce the decomposition

$$\mathbf{J}' = \mathbf{J} + a\mathbf{M}, \tag{1.7}$$

in order to have all the eigenvalues included in the circle with radius one in the complex plane. The two theorems shown in this paper describe in detail the stability and asymptotic properties of the dynamic associated to the matrix $\mathbf{J}'$ and in particular these properties are very sensible to the choice of the initial conditions for the $x_i(t)$. We give here some hint since the complete definitions will be given in the next section. So suppose that the initial conditions can be written in the following way:

$$x_i(0) = c_i + \xi_i, \tag{1.8}$$

where $\{\xi_i\}$ are independent random variables with distributions $\{\nu_i\}$, satisfying the conditions

$$E\{\xi_i\} = 0, \quad E\{(\xi_i)^2\} = \sigma_i^{(0)}, \quad E\{(\xi_i)^8\} \le C. \tag{1.9}$$

and the initial constants depend on the neuron in the same way as the

$$c_i = \left\{ \begin{array}{ll} c_I, & i = 1, \dots [fn], \\ c_E, & otherwise, \end{array} \right. \tag{1.10}$$

In this situation the theorems proved in the paper establish that the contribution to the dynamic of the matrix $\mathbf{J}$ is stable if $\kappa > \sigma_* = f\sigma_I + (1-f)\sigma_E$. Remark that since the matrix $\mathbf{J}$ is asymmetric the stability of the dynamic does not depend on the maximum eigenvalue so it is reasonable that the stability results for the dynamic generated by this matrix is different from the one enunciated above. The matrix $\mathbf{M}$ also contributes to the dynamic and, due to its particular form, gives some unexpected result, namely if $c_I \neq c_E$ the average of the contribution of $\mathbf{M}$ to the dynamic goes

like $t\sqrt{n}$ so it is divergent for large n but it goes in any case to zero due to the multiplication with the exponential $e^{-\kappa t}$. Thus we expect in this case to have large coherent motions which are then dumped by the exponential factor. If $c_I = c_E$ the situation is completely different because the term of the order $t\sqrt{n}$ is multiplied by a constant equal to zero and disappears. In this case the contribution of $\mathbf{M}$ converges to a gaussian random variable with zero mean and variance of the type of a constant $+J_0(\sqrt{\sigma}_* t)$ and so we get the usual stability result. These are the meaning of the theorems demonstrated in the paper.

2 Notations and formulations of the results

Consider a dynamical system with random interactions (so-called a complex system in [9]) defined by

$$\mathbf{x}' = -\kappa\mathbf{x} + \mathbf{J}'\mathbf{x}, \tag{2.1}$$

where $\mathbf{x} \in \mathbb{R}^n$, κ is a real number and $\mathbf{J}'$ is an $n \times n$ real random matrix. Following Rajan and Abbott (see ([10])) we consider the case, when J_{ij} are independent Gaussian variables with mean values

$$E\{J'_{ij}\} = a \cdot \begin{cases} \mu_I/n^{1/2}, & j = 1, \dots [fn], \\ \mu_E/n^{1/2}, & otherwise. \end{cases} \tag{2.2}$$

Here $0 < f < 1$ and $a > 0$ are fixed parameters, and μ_I and μ_E are chosen so that the vector $\mathbf{m} = (m_1, \dots, m_n)$

$$m_i = \begin{cases} \mu_I/n^{1/2}, & j = 1, \dots [fn] \\ \mu_E/n^{1/2} & , otherwise, \end{cases} \tag{2.3}$$

satisfies conditions

$$(\mathbf{m}, \mathbf{u}) = 0, \quad (\mathbf{m}, \mathbf{m}) = 1 \tag{2.4}$$

with

$$\mathbf{u} = (1, \dots, 1). \tag{2.5}$$

The variances of J_{ij} are chosen as follows

$$E\{(J'_{ij} - E\{J'_{ij}\})^2\} = n^{-1}\sigma_j = \begin{cases} \sigma_I/n, & j = 1, \dots [fn], \\ \sigma_E/n, & otherwise. \end{cases} \tag{2.6}$$

It is easy to see that in this case the matrix J' could be represented in the form

$$\mathbf{J}' = \mathbf{J} + a\mathbf{M}, \tag{2.7}$$

where the matrix $\mathbf{M}$ is a rang one matrix all rows equal to $\mathbf{m}$, so that

$$\mathbf{M}\mathbf{x} = (\mathbf{m}, \mathbf{x})\mathbf{u}, \quad \mathbf{x} \in \mathbb{R}^n, \tag{2.8}$$

and the matrix $\mathbf{J}$ has the form

$$\mathbf{J} = n^{-1/2}\mathbf{W} \tag{2.9}$$

where $\mathbf{W}$ a Gaussian matrix with independent entries satisfying conditions

$$E\{W_{ij}\} = 0, \quad E\{W_{ij}^2\} = \sigma_j \tag{2.10}$$

with σ_j defined in (2.6).

As it was shown numerically in the paper [10], the matrix $\mathbf{J}'$ under conditions (2.2)-(2.6) has a spectrum which is not localized in some fixed domain of $\mathbb{C}$ and so it is difficult to expect that

the dynamics of the system (2.1) will be stable. But if we introduce the additional equilibrium conditions

$$\mathbf{J}\mathbf{u} = 0 \Leftrightarrow \sum_{j=1}^{n} W_{ij} = 0 \quad (i = 1, \ldots, n), \tag{2.11}$$

then the spectrum $\mathbf{J}'$ coincides with the spectrum of $\mathbf{J}$, which is well localized according to the results of [**?**, **?**].

It is easy to see, that under conditions (2.4), (2.8) and (2.11)

$$\mathbf{M}^2 = 0, \quad \mathbf{J}\mathbf{M} = 0 \tag{2.12}$$

so that for any $k \geq 1$

$$(\mathbf{J} + a\mathbf{M})^k = \mathbf{J}^k + a\mathbf{M}\mathbf{J}^{k-1}.$$

Hence

$$e^{t(\mathbf{J}+\mathbf{M})} = e^{t\mathbf{J}} + a \int_0^t ds \mathbf{M} e^{s\mathbf{J}} \tag{2.13}$$

and the solution of the system (2.1) could be represented in the form

$$x_i(t) = e^{-\kappa t}(e^{t\mathbf{J}}\mathbf{x}(0))_i + ae^{-\kappa t} \int_0^t ds(e^{s\mathbf{J}}\mathbf{x}(0), \mathbf{m}), \tag{2.14}$$

where $\mathbf{x}(0)$ is a vector of initial conditions. Thus to study the dynamics (2.1) it suffices to study the dynamics of the system

$$\mathbf{x}' = \mathbf{J}\mathbf{x} \tag{2.15}$$

with a matrix $\mathbf{J}$ of the form (2.9), and $\mathbf{W}$, satisfying conditions (2.10) and (2.11).

Supply the system with the initial conditions

$$x_i(0) = c_i + \xi_i, \tag{2.16}$$

where $\{\xi_i\}$ are independent random variables with distributions $\{\nu_i\}$, satisfying the conditions

$$E\{\xi_i\} = 0, \quad E\{(\xi_i)^2\} = \sigma_i^{(0)}, \quad E\{(\xi_i)^8\} \leq C. \tag{2.17}$$

and

$$c_i = \begin{cases} c_I, & i = 1, \ldots [fn], \\ c_E, & otherwise, \end{cases} \qquad \nu_i(x) = \begin{cases} \nu_I(x), & i = 1, \ldots [fn], \\ \nu_E(x), & otherwise, \end{cases} \qquad \sigma_i^{(0)} = \begin{cases} \sigma_I^{(0)}, & i = 1, \ldots [fn], \\ \sigma_E^{(0)}, & otherwise. \end{cases} \tag{2.18}$$

Define the normalized counting function of x_i, solutions of the system (2.15),

$$\mathcal{N}_n(\lambda, t) = n^{-1} \sharp\{x_i(t) \leq \lambda\} = n^{-1} \sum_{i=1}^{n} \theta(\lambda - x_i(t)), \tag{2.19}$$

where $\theta(x)$ is the standard Heaviside function. $\mathcal{N}_n(\lambda, t)$ is a random measure on the real line which counts the fraction of the variables $x_1, \ldots, x_n$ which are less then λ at time t. Thus it characterizes the distribution of $x_i(t)$ on the real line.

Theorem 1 *Consider the system (2.15) with a matrix* $\mathbf{J}$ *of the form (2.9) under conditions (2.10) and (2.11), and supply this system by the initial conditions (2.16)-(2.18). Then for any* $t > 0$*,*

$$\lim_{n \to \infty} \mathcal{N}_n(\lambda, t) = f\mathcal{N}_I(\lambda, t) + (1 - f)\mathcal{N}_E(\lambda, t) \tag{2.20}$$

where $\mathcal{N}_I(\lambda,t)$ *and* $\mathcal{N}_E(\lambda,t)$ *are the convolutions of the initial distribution* ν_I *and* ν_E *with normal distributions* $\mathcal{N}(c_I,\tilde\sigma(t))$ *and* $\mathcal{N}(c_E,\tilde\sigma(t)$ *respectively*

$$\mathcal{N}_I(\lambda,t)=\left(\nu_I*\mathcal{N}(c_I,\tilde\sigma(t))\right)(\lambda),\quad \mathcal{N}_E(\lambda,t)=\left(\nu_E*\mathcal{N}(c_E,\tilde\sigma(t))\right)(\lambda),\tag{2.21}$$

and the variance $\tilde\sigma(t)$ *has the form*

$$\tilde\sigma(t)=A\sigma_*^{-1}\sum_{m=1}^{\infty}\frac{\sigma_*^m t^{2m}}{m!m!}.\tag{2.22}$$

where

$$\begin{aligned}&\sigma_*=f\sigma_I+(1-f)\sigma_E,\\ &A=\sigma_*^{-1}\sigma_I\sigma_E f(1-f)(c_I-c_E)^2+(\sigma_I\sigma_I^{(0)}f+\sigma_E\sigma_E^{(0)}(1-f)).\end{aligned}\tag{2.23}$$

Theorem 2 *Consider the system (2.15) with matrix* $\mathbf{J}$ *of the form (2.9) under conditions (2.10) and (2.11), and supply this system by the initial conditions (2.16) with (2.17). Set*

$$w_n(t)=\int_0^t ds(e^{s\mathbf{J}}\mathbf{x}(0),\mathbf{m}).\tag{2.24}$$

If $c_I\neq c_E$*, then*

$$E\{w_n(t)\}=n^{1/2}t(fc_I\mu_I+(1-f)c_E\mu_E)\sim\sqrt{n}.\tag{2.25}$$

If $c_I=c_E$*, then* $w_n(t)$ *for each fixed* t *converges in distribution to a Gaussian random variable with zero mean and variance*

$$\tilde\sigma^{(0)}=(1-f)\sigma_I^{(0)}+f\sigma_E^{(0)}+A\sigma_*^{-1}\sum_{m=1}^{\infty}\frac{\sigma_*^m t^{2m}}{m!m!},\tag{2.26}$$

where A *and* σ_* *are defined in (2.23).*

3 Proofs

First of all we need to compute the expectations of $E\{W_{ij}\}$ and $E\{W_{ij}W_{kl}\}$ under conditions (2.11)

Lemma 1 *(i) Under conditions (2.11) and (2.10)*

$$E\{W_{ij}\}=0,\quad E\{W_{ij}W_{kl}\}=\delta_{ik}\left(\delta_{jl}\sigma_j-\frac{\sigma_j\sigma_l}{n\sigma_*}\right),\tag{3.1}$$

where

$$\sigma_*=n^{-1}\sum\sigma_k=n^{-1}([fn]\sigma_I+(n-[fn])\sigma_E).\tag{3.2}$$

(ii) Consider the random variable of the form

$$z=\sum_{k=1}^{n}n^{-1/2}W_{1k}d_k,\tag{3.3}$$

where the coefficients d_k *do not depend on* $\{W_{1k}\}$*. Then* z *is a normal variable with zero mean and the variance*

$$\sigma_z=n^{-1}\sum d_k^2\sigma_k-\sigma_*^{-1}(n^{-1}\sum d_k\sigma_k)^2\tag{3.4}$$

Proof of Lemma 1 The first equality in (3.1 is evident, because conditions (2.11) are symmetric with respect to the change $W_{ik} \to -W_{ik}$. Besides, since different lines of the matrix $\mathbf{W}$ have independent entries it is evident that for $i \neq k$ $E\{W_{ij}W_{kl}\} = 0$, and for $i = k$ $E\{W_{kj}W_{kl}\}$ do not depend on k. Hence,

$$E\{W_{kj}W_{kl}\} = E\{W_{1j}W_{1l}\} = \lim_{\varepsilon\to 0} \frac{\int W_{1j}W_{1l}\exp\{-\sum W_{1j}^2/2\sigma_j - (\sum W_{1j})^2/(2n\varepsilon)\}d\mathbf{W}}{\int \exp\{-\sum W_{1j}^2/2\sigma_j - (\sum W_{1j})^2/(2n\varepsilon)\}d\mathbf{W}}$$

$$= \lim_{\varepsilon\to 0} \frac{\int dW_{1j}W_{1l}\exp\{-\sum W_{1j}^2/2\sigma_j + it\sum n^{-1/2}W_{1j} - \varepsilon t^2/2\}d\mathbf{W}dt}{\int \exp\{-\sum W_{1j}^2/2\sigma_j + it\sum n^{-1/2}W_{1j} - \varepsilon t^2/2\}d\mathbf{W}dt} = \quad (3.5)$$

$$= \frac{\int (\delta_{jl}\sigma_j - n^{-1}t^2\sigma_j\sigma_l)\exp\{-t^2\sum \sigma_j/(2n)\}dt}{\int \exp\{-t^2\sum \sigma_j/(2n)\}dt} =$$

$$= \delta_{jl}\sigma_j - \frac{\sigma_j\sigma_l}{n\sigma_*},$$

where $d\mathbf{W} = \prod_{j=1}^{n} dW_{kj}$.

To prove the assertion (ii) of Lemma 1 we compute by the same way the characteristic function of z

$$E\{e^{isz}\} = \lim_{\varepsilon\to 0} \frac{\int \exp\{-\sum W_{1j}^2/2\sigma_j + is\sum_{k=1}^n n^{-1/2}W_{1k}d_k - (\sum W_{1j})^2/(2n\varepsilon)\}d\mathbf{W}}{\int \exp\{-\sum W_{1j}^2/2\sigma_j - (\sum W_{1j})^2/(2n\varepsilon)\}d\mathbf{W}}$$

$$= \lim_{\varepsilon\to 0} \frac{\int \exp\{-\sum W_{1j}^2/2\sigma_j + is\sum_{k=1}^n n^{-1/2}W_{1k}d_k + it\sum n^{-1/2}W_{1j} - \varepsilon t^2/2\}d\mathbf{W}dt}{\int \exp\{-\sum W_{1j}^2/2\sigma_j + it\sum n^{-1/2}W_{1j} - \varepsilon t^2/2\}d\mathbf{W}dt} = \quad (3.6)$$

$$= \frac{\int \exp\{-\sum (t+sd_k)^2\sigma_j/(2n)\}dt}{\int \exp\{-t^2\sum \sigma_j/(2n)\}dt} = e^{-s^2\sigma_z/2}.$$

Lemma 1 is proved.

Below it will be convenient to consider the matrix $\mathbf{J}$ in the new orthonormal basis. Denote

$$E_1 = \mathrm{Lin}\{\mathbf{e}_1, \dots, \mathbf{e}_{[fn]}\}, \quad E_2 = \mathrm{Lin}\{\mathbf{e}_{[fn]+1}, \dots, \mathbf{e}_n\}, \qquad (3.7)$$

where $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the basis in which we consider the system (2.15) initially, so that $x_i = (\mathbf{x}, \mathbf{e}_i)$.

Then define in E_1 and E_2 the orthonormal systems $\{\mathbf{u}_3, \dots, \mathbf{u}_{[fn]+1}\}$ and $\{\mathbf{u}_{[fn]+2}, \dots, \mathbf{u}_n\}$ which are orthogonal to the vectors $\mathbf{e}_1 + \dots + \mathbf{e}_{[fn]}$, and $\mathbf{e}_{[fn]+1} + \dots + \mathbf{e}_n$ respectively. If we denote

$$\mathbf{u}_1 = n^{-1/2}\mathbf{u}, \quad \mathbf{u}_2 = m, \qquad (3.8)$$

then, according to (2.5), (2.4) and our choice of $\mathbf{u}_3, \dots, \mathbf{u}_n$, the system $\{\mathbf{u}_i\}_{i=1}^n$ forms an orthonormal basis in $\mathbb{R}^n$. Let $(u_{1i}, \dots, u_{ni})$ be the components of the vector $\mathbf{u}_i$ in the basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. Then the matrix

$$\mathbf{U} = \{u_{ki}\}_{k,i=1}^n \qquad (3.9)$$

is a matrix of the orthogonal transformation from the basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ to the basis $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$. Consider the matrix $\mathbf{J}$ in this basis.

$$\tilde{\mathbf{J}} = \mathbf{U}^*\mathbf{J}\mathbf{U} = n^{-1/2}\mathbf{U}^*\mathbf{W}\mathbf{U} = n^{-1/2}\tilde{\mathbf{W}}. \qquad (3.10)$$

Lemma 2 *The entries* $\{\tilde{W}_{ki}\}_{k,i=1}^n$ *are independent Gaussian variables with zero means and their variances are*

$$E\{\tilde{W}_{ij}^2\} = \begin{cases} 0, & j = 1, \\ \sigma_I, & j = 3, \dots [fn]+1, \\ \sigma_E, & j = [fn]+2\dots,n, \\ \sigma_I\sigma_E/\sigma_*, & j = 2. \end{cases} \tag{3.11}$$

Remark 1 *It follows from Lemma 2 that the matrix* $\tilde{\mathbf{J}}$ *can be represented in the form*

$$\tilde{\mathbf{J}} = \tilde{\mathbf{J}}_1\mathbf{D}^{(1)},$$

where $\tilde{\mathbf{J}}_1$ *is a matrix with i.i.d. entries Gaussian entries with zero means and variances* 1, *and* $\mathbf{D}^{(1)}$ *is a diagonal matrix with* $D_{11}^{(1)} = 0$, $D_{22}^{(1)} = (\sigma_I\sigma_E/\sigma_*)^{1/2}$ $D_{ii}^{(1)} = \sigma_I^{1/2}$ *for* $i = 3, \dots [fn]+1$ *and* $D_{ii}^{(1)} = \sigma_E^{1/2}$ *for* $i = [fn]+2\dots,n$. *Hence*

$$||\tilde{\mathbf{J}}||^2 \le \max\{\sigma_I, \sigma_E\}||\tilde{\mathbf{J}}_1||^2.$$

*Using the result of [**?**], according to which under condition matrix* $\tilde{\mathbf{J}}_1$ *with i.i.d.*

$$Prob\{||\tilde{\mathbf{J}}_1||^2 > 4+\varepsilon\} \le e^{-C_1 n\varepsilon^2},$$

and the fact that $||\tilde{J}|| = ||\mathbf{J}||$, *we obtain now that*

$$Prob\{||\mathbf{J}|| > 2L+\varepsilon\} \le e^{-Cn\varepsilon^2}, \tag{3.12}$$

where we denote

$$L = \max\{\sigma_I^{1/2}, \sigma_E^{1/2}\}. \tag{3.13}$$

Remark 2 *Inequality (3.12) allows us to use* $||\mathbf{J}||$ *in our considerations like a bounded random variables. Indeed, since, e.g.,* $|x_1(t)| \le ne^{t||\mathbf{J}||}$, *denoting* $P_n(\lambda) = \text{Prob}\{||\mathbf{J}|| > 2L+\lambda\}$ *and using (3.12), we can write for any fixed* t *and* $m, s << n/\log n$

$$E\{|x_1(t)|^m e^{s||\mathbf{J}||}\} \le$$
$$\le e^{s(2L+\epsilon)}E\{|x_1(t)|^m\theta(2L+\epsilon-||\mathbf{J}||)\} + n^m E\{e^{(s+mt)||\mathbf{J}||}\theta(||\mathbf{J}||-2L-2\epsilon)\} \le$$
$$\le e^{s(2L+\epsilon)}E\{|x_1(t)|^m\} + n^m\int_{\lambda>\epsilon} e^{(s+mt)\lambda}dP_n(\lambda) \le e^{s(2L+\epsilon)}E\{|x_1(t)|^m\} + O(e^{-Cn\varepsilon^2/2}).$$

Hence, below we use $||\mathbf{J}||$ *as a bounded variable without additional explanations.*

Proof of Lemma 2. It is evident that $\{\tilde{W}_{ki}\}_{k,i=1}^n$ have joint Gaussian distribution, so to prove Lemma 2 it is enough to compute their covariances. To this aim we use relation

$$\tilde{W}_{ij} = \sum_{k,l} u_{ki}W_{kl}u_{lj} \tag{3.14}$$

and Lemma 1. Then from the first equality of 3.1 we derive that the mean values of $\{\tilde{W}_{ki}\}_{k,i=1}^n$ are equal to zero.

$$E\{\tilde{W}_{i_1j_1}\tilde{W}_{i_2j_2}\} = \sum u_{k_1i_1}u_{l_1j_1}u_{k_2i_2}u_{l_2j_2}E\{W_{k_1l_1}W_{k_2l_2}\}. \tag{3.15}$$

Now we use the fact that for different k_1 and k_2 $W_{k_1l_1}$ $W_{k_2l_2}$ are independent and for $k_1 = k_2$ $E\{W_{k_1l_1}W_{k_1l_2}\}$ does not depend on k_1 (see (3.1)). Substituting 3.1) in (3.15), summing with respect to k_1 and using the orthogonality of $\mathbf{u}_{i_1}$ and $\mathbf{u}_{i_2}$, we get

$$E\{\tilde{W}_{i_1j_1}\tilde{W}_{i_2j_2}\} = \delta_{i_1i_2}\sum u_{l_1j_1}u_{l_2j_2}(\delta_{l_1l_2}\sigma_{l_1} - \sigma_{l_1}\sigma_{l_2}/(\sigma_* n)). \tag{3.16}$$

Now if $j_1 = 1$, then $u_{l_1j_1} = n^{-1/2}$ and summation with respect to l_1 gives us zero because of (3.2). If $j_1 = 3, \dots, [fn]+1$, then, since $u_{l_1j_1} = 0$ for $l_1 \geq [fn]+1$, we have that in the r.h.s. of (3.16) $\sigma_{l_1} = \sigma_I$, and so, using the orthogonality of $\mathbf{u}_{j_1}$ and $\mathbf{u}_{j_2}$, we get the second line of (3.11). If $j_1 = [fn]+2, \dots, n$ the proof is the same. Now we are left to prove the last line of (3.11). Using (3.16) and (2.4), which gives us

$$\mu_I = -\sqrt{(n-[fn])/[fn]}, \quad \mu_E = \sqrt{[fn]/(n-[fn])}, \tag{3.17}$$

we obtain

$$E\{\tilde{W}_{i2}\tilde{W}_{i2}\} \;=\; n^{-1}\sum_{l_1=1}^{[fn]}\sigma_I\mu_I^2 + n^{-1}\sum_{l_1=[fn]+1}^{n}\sigma_E\mu_E^2 - (\sigma_* n^2)^{-1}\left(\sum_{l_1=1}^{[fn]}\sigma_I\mu_I + \sum_{[fn]+1}^{n}\sigma_E\mu_E\right)^2 = \tag{3.18}$$

$$= (\sigma_I(1-[fn]/n) + \sigma_E[fn]/n) - \frac{(\sigma_I-\sigma_E)^2[fn](n-[fn])}{\sigma_* n^2} =$$

$$= \frac{\sigma_I\sigma_E}{\sigma_*}.$$

Proof of Theorem 1 Let us consider the system (2.15) from the second equation to the last one as a system of equations for $x_2(1), \dots, x_n(t)$, where $x_1(t)$ is a known function. Then

$$x_i(t) = x_i^{(1)} + \sum_{j=2}^{n}\int_0^t ds (e^{(t-s)\mathbf{J}^{(1)}})_{ij}\frac{W_{j1}}{n^{1/2}}x_1(s), \tag{3.19}$$

where

$$x_i^{(1)} = (e^{t\mathbf{J}^{(1)}}\mathbf{x}(0))_i \tag{3.20}$$

and $\mathbf{J}^{(1)}$ is the matrix which we obtain from $\mathbf{J}$ replacing the first line and the first column by zeros. Substituting this expressions in the first equation of (2.15), we get

$$x_1'(t) = \sum_{j=2}^{n}\frac{W_{1j}}{n^{1/2}}x_j^{(1)}(t) + \int_0^t ds\, \tilde{r}_n^{(1)}(t-s)x_1(s) + \frac{W_{11}}{n^{1/2}}x_1(t), \tag{3.21}$$

where

$$\tilde{r}_n^{(1)}(t) = n^{-1}\sum_{i,j=2}^{n}(e^{t\mathbf{J}^{(1)}})_{ij}W_{1i}W_{j1}. \tag{3.22}$$

Hence

$$x_1(t) = x_1(0) + \sum_{j=2}^{n}\frac{W_{1j}}{n^{1/2}}d_j^{(1)}(t) + \int_0^t ds\, r_n^{(1)}(t-s)x_1(s), \tag{3.23}$$

with

$$d_j^{(1)}(t) = \int_0^t ds\, x_j^{(1)}(s), \quad r_n^{(1)}(t) = \int_0^t d\tau\, \tilde{r}_n^{(1)}(\tau) + n^{-1/2}W_{11}.$$

Using Lemma 1, it is easy to see that

$$E\{(\tilde{r}_n^{(1)}(t))^8\} \le C(t)n^{-4}. \tag{3.24}$$

Indeed, according to (3.22),

$$\tilde{r}_n^{(1)} = n^{-1/2}\sum_{i=2}^{n} W_{1i} f_i, \quad f_i = n^{-1/2}\sum_{j=2}^{n}(e^{t\mathbf{J}^{(1)}})_{ij}W_{j1}$$

with f_i independent of $\{W_{1j}\}$. Hence, if we take the eighth power of (3.22) and take the expectation with respect to $\{W_{1j}\}$, we get

$$E\{(\tilde{r}_n^{(1)}(t))^8\} = \tag{3.25}$$

$$= 7\cdot 5\cdot 3\cdot E\left\{\left(n^{-2}\sum_{j_1,j_2}(e^{t\mathbf{J}^{(1)T}}\mathbf{D}e^{t\mathbf{J}^{(1)}})_{j_1j_2}W_{j_11}W_{j_21} - \; - \; n^{-3}\sigma_*^{-1}(\sum_{i_1,j_1}\sigma_i(e^{t\mathbf{J}^{(1)}})_{i_1j_1}W_{j_11})^2\right)^4\right\} \le$$

$$\le 105 n^{-4} E\left\{||D||e^{8t||J^{(1)}||}\left(n^{-1}\sum_{j_1}W_{j_11}^2\right)^2\right\} \le$$

$$\le C(t)n^{-4},$$

where $\mathbf{J}^{(1)T}$ means the transposed matrix of $\mathbf{J}^{(1)T}$, $\mathbf{D}$ is a diagonal matrix such that

$$D_{ij} = \delta_{ij}\sigma_i, \tag{3.26}$$

and here and below we denote by $C(t)$ function of t (different in different formulas), such that

$$C(t) \le Ce^{ct}$$

with some n-independent C and c.

The relation (3.25) and a trivial crude bound

$$|x_1(t)| \le ||e^{t\mathbf{J}}\mathbf{x}(0)|| \le e^{t||\mathbf{J}||}||x(0)|| \le C(t)\sqrt{n}$$

allow us to obtain

$$E\{x_1^4(t)\} \le \tag{3.27}$$

$$\le 27\left(E\{x_1^4(0)\} + 3L^2\left(n^{-1}\sum(d_i^{(1)}(t))^2\right)^2 \; + t^3\int_0^t dsE^{1/2}\{(r_n^{(1)}(t-s))^8\}E^{1/2}\{x_1^8(s)\}\right) \le$$

$$\le 27\left(C + 3L^2E\{(n^{-1}||e^{t\mathbf{J}^{(1)}}\mathbf{x}(0)||^2)\} + C(t)n^{-4}\int_0^t dsE^{1/2}\{x_1^4(s)e^{4s||\mathbf{J}||}||x(0)||^4\}\right) \le$$

$$\le C(t)\left(1 + n^{-2}\int_0^t dsE^{1/2}\{x_1^4(s)\}\right),$$

where L is defined by (3.13). Then, by a standard argument, we get

$$E\{x_1^4(t)\} \le C(t). \tag{3.28}$$

This bound allows us to write (3.23) as

$$x_1(t) = x_1(0) + \sum_{j=2}^{n}\frac{W_{1j}}{n^{1/2}}d_j^{(1)}(t) + \varepsilon_n^{(1)}(t), \tag{3.29}$$

where

$$E\{(\varepsilon_n^{(1)}(t))^2\} \le C(t)n^{-1}. \tag{3.30}$$

Now we can apply Lemma 1, which gives us that the sum in the r.h.s. of (3.29) is a normal random variable with the variance

$$\tilde{\sigma}_n^{(1)}(t) = n^{-1}\sum \sigma_j (d_j^{(1)}(t))^2 - \sigma_*^{-1}(n^{-1}\sum \sigma_i d_i^{(1)})^2. \tag{3.31}$$

Define

$$R_n(t,s) = n^{-1}\sum_{j=1}^n \sigma_j x_j(t)x_j(s). \tag{3.32}$$

Lemma 3 *Under conditions of Theorem 1 $R_n(t,s)$*

$$D_n(t,s) = E\{(R_n(t,s) - E\{R_n(t,s)\})^2\} \le C(t)C(s)n^{-1}, \tag{3.33}$$

$$E\Big\{\Big(n^{-1}\sum_{i=1}^n \sigma_i x_i(t) - E\Big\{n^{-1}\sum_{i=1}^n \sigma_i x_i(t)\Big\}\Big)^2\Big\} \le C(t)n^{-1}. \tag{3.34}$$

Besides,

$$E\{|R_n(t,s) - n^{-1}\sum_{j=1}^n \sigma_j x_j^{(1)}(t)x_j^{(1)}(s)|^2\} \le C(t)C(s)n^{-1}. \tag{3.35}$$

Denote

$$\tilde{\sigma}_n(t) = \int_0^t\int_0^t dt'ds' R_n(t',s') - \sigma_*^{-1}(\int_0^t dt' n^{-1}\sum \mathbf{x}_i(t'))^2. \tag{3.36}$$

Remark 3 *Lemma 3 implies*

$$E\{(\tilde{\sigma}_n^{(1)}(t) - \tilde{\sigma}_n(t))^2\} \le C(t)n^{-1}, \tag{3.37}$$

$$E\{(\tilde{\sigma}_n(t) - E\{\tilde{\sigma}_n(t)\})^2\} \le C(t)n^{-1}. \tag{3.38}$$

Proof of Lemma 3. To prove (3.35) we first estimate

$$n^{-1}\sum_{j=1}^n \sigma_j(x_j(t) - x_j^{(1)}(t))(x_j(s) - x_j^{(1)}(s)) = \tag{3.39}$$

$$= n^{-1}\sum_{j_1,j_2=2}^n \int_0^s ds_1 \int_0^t ds_2 (e^{(t-s_1)\mathbf{J}^{(1)T}}\mathbf{D}e^{(s-s_2)\mathbf{J}^{(1)}})_{j_1j_2}\frac{W_{1j_1}W_{1j_1}}{n} x_1(s_1)x_1(s_2) \le$$

$$\le n^{-2}tse^{(s+t)||\mathbf{J}^{(1)}||}\left(\int_0^t x_1^2(s_1)ds_1\right)^{1/2}\left(\int_0^s x_1^2(s_2)ds_2\right)^{1/2}\sum_{j=2}^n W_{j1}^2.$$

Now, using the Schwartz inequality we get

$$\left|R_n(t,s) - n^{-1}\sum_{j=2}^n \sigma_j x_j^{(1)}(t)x_j^{(1)}(s)\right|^2 = \left|n^{-1}\sum_{j=1}^n \sigma_j\left((x_j(t) - x_j^{(1)}(t))x_j(s) + x_j^{(1)}(t)(x_j(s) - x_j^{(1)}(s))\right)\right|^2$$

$$\le \left(n^{-1}\sum_{j=1}^n \sigma_j\left((x_j(t) - x_j^{(1)}(t))^2 + (x_j(s) - x_j^{(1)}(s))^2\right)\right)\left(n^{-1}\sum_{j=1}^n \sigma_j\left(x_j^2(s) + (x_j^{(1)}(t))^2\right)\right)$$

$$\le Cn^{-3}tse^{2(s+t)||\mathbf{J}^{(1)}||}\left(\int_0^t x_1^2(s_1)ds_1 + \int_0^s x_1^2(s_2)ds_2\right)||x(0)||^2\sum_{j=2}^n W_{j1}^2. \tag{3.40}$$

Combining this with (3.28) we obtain (3.35). To prove (3.33), we write

$$D_n(t,s)=\sum_{i,j=1}^n \sigma_i\sigma_j E\Big\{\Big(x_i(t)x_i(s)-E\{x_i(t)x_i(s)\}\Big)\Big(x_j(t)x_j(s)-E\{x_j(t)x_j(s)\}\Big)\Big\}. \qquad (3.41)$$

Let us estimate, e.g. $E\Big\{\Big(x_1(t)x_1(s)-E\{x_1(t)x_1(s)\}\Big)\Big(x_2(t)x_2(s)-E\{x_2(t)x_2(s)\}\Big)\Big\}$. To this end we write the representations (cf.(3.23))

$$\begin{aligned} x_1(t) &= x_1(0)+n^{-1/2}W_{12}d_2(t)+n^{-1/2}\sum_{j=3}^n W_{1j}d_j^{(1,2)}(t)\\ &\quad+\int_0^t dt'\,r_n^{(1,1)}(t-t')x_1(t')+\int_0^t dt'\,r_n^{(1,2)}(t-t')x_2(t'),\\ x_2(t) &= x_2(0)+n^{-1/2}W_{21}d_1(t)+n^{-1/2}\sum_{j=3}^n W_{2j}d_j^{(1,2)}(t)\\ &\quad+\int_0^t dt'\,r_n^{(2,1)}(t-t')x_1(t')+\int_0^t dt'\,r_n^{(2,2)}(t-t')x_2(t'), \end{aligned} \qquad (3.42)$$

where

$$d_j^{(1,2)}(t)=\int_0^t ds\,x_j^{(1,2)}(s),\quad x_j^{(1,2)}(t)=(e^{t\mathbf{J}^{(1,2)}}\mathbf{x}(0))_j,\quad r_n^{(\alpha,\beta)}(t)=\int_0^t d\tau\,\tilde r_n^{(\alpha,\beta)}(\tau),$$

$$\tilde r_n^{(\alpha,\beta)}(t)=n^{-1}\sum_{i,j=2}^n (e^{t\mathbf{J}^{(1,2)}})_{ij}W_{\alpha i}W_{j\beta}.$$

$\mathbf{J}^{(1,2)}$ is the matrix $\mathbf{J}$ with the first and the second lines and the first and the second columns replaced by zeros. Similarly to (3.24)-(3.30) we obtain

$$\begin{aligned} x_1(t) &= x_1(0)+n^{-1/2}\sum_{j=3}^n W_{1j}d_j^{(1,2)}(t)\;+\varepsilon_n^{(1)}(t),\\ x_2(t) &= x_2(0)+n^{-1/2}\sum_{j=3}^n W_{2j}d_j^{(1,2)}(t)+\varepsilon_n^{(2)}(t),\\ &\quad E\{(\varepsilon_n^{(1)}(t))\}\le n^{-1}C(t),\ E\{(r_n^{(2)}(t))\}\le n^{-1}C(t). \end{aligned} \qquad (3.43)$$

Then we can write

$$\begin{aligned} &E\Big\{\Big(x_1(t)x_1(s)-E\{x_1(t)x_1(s)\}\Big)\Big(x_2(t)x_2(s)-E\{x_2(t)x_2(s)\}\Big)\Big\}\\ &\quad\le\int_0^t\int_0^s\int_0^t\int_0^s dt_1'ds_1'dt_2'ds_2'E\Big\{\Big(R^{(1,2)}(t_1',s_1')-E\{R^{(1,2)}(t_1',s_1')\}\Big)\\ &\qquad\qquad\Big(R^{(1,2)}(t_2',s_2')-E\{R^{(1,2)}(t_2',s_2')\}\Big)\Big\}+C(t)C(s)n^{-1}\\ &\quad\le\int_0^t\int_0^s\int_0^t\int_0^s dt_1'ds_1'dt_2'ds_2'\Big(E\Big\{\Big(R^{(1,2)}(t_1',s_1')-E\{R^{(1,2)}(t_1',s_1')\}\Big)^2\Big\}\\ &\qquad+E\Big\{\Big(R^{(1,2)}(t_2',s_2')-E\{R^{(1,2)}(t_2',s_2')\}\Big)^2\Big\}\Big)+C(t)C(s)n^{-1}\\ &\quad\le 2ts\int_0^t\int_0^s dt'ds'E\Big\{\Big(R^{(1,2)}(t',s')-E\{R^{(1,2)}(t',s')\}\Big)^2\Big\}+C(t)C(s)n^{-1}, \end{aligned} \qquad (3.44)$$

where

$$R^{(1,2)}(t,s) = n^{-1}\sum_{i=3}^{n}\sigma_i x_i^{(1,2)}(t)x_i^{(1,2)}(s).$$

Similarly to (3.35)

$$E\{(R_n^{(1,2)}(t,s) - R_n(t,s))^2\} \le C(t)C(s)n^{-1}.$$

Repeating (3.44) for all terms in (3.41) with different i,j, we obtain the inequality

$$D_n(t,s) \le 2\sigma_*^2 ts\int_0^t\int_0^s dt'ds' D_n(t',s').$$

Iterating this inequality $M = [\log n]$ times, we get (3.33). The proof of the inequality (3.34) follows from the representation (3.42) immediately, if we use the independence of $x_1(0)$ and $x_2(0)$.

Lemma 3 is proved.

Now we are ready to prove the self averaging property of $\mathcal{N}_n(\lambda,t)$, as $n\to\infty$, i.e. we prove that for any real λ and $t>0$

$$\lim_{n\to\infty} E\left\{\left(\mathcal{N}_n(\lambda,t) - E\{\mathcal{N}_n(\lambda,t)\}\right)^2\right\} = 0. \tag{3.45}$$

According to the standard theory of measure, for this aim it is enough to prove that $g_n(z,t)$ – the Stieltjes transform of the distribution $\mathcal{N}_n(\lambda,t)$

$$g_n(z,t) = \int\frac{d\mathcal{N}_n(\lambda,t)}{\lambda - z} = n^{-1}\sum_{i=1}^{n}\frac{1}{x_i(t)-z}, \quad (\Im z\ne 0), \tag{3.46}$$

for any $z:\Im z\ne 0$ possesses a self averaging. property.

Lemma 4 *For any $z:\Im z\ne 0$*

$$\lim_{n\to\infty} E\left\{\left|g_n(z,t) - E\{g_n(z,t)\}\right|^2\right\} = 0. \tag{3.47}$$

Proof of Lemma 4. Similarly to (3.41) we write

$$\begin{aligned} &E\left\{\left|g_n(z,t) - E\{g_n(z,t)\}\right|^2\right\} \\ &\quad = n^{-2}\sum_{i,j=1}^{n}\left(E\left\{(x_i(t)-z)^{-1}(x_j(t)-\overline{z})^{-1}\right\} - E\left\{(x_i(t)-z)^{-1}\right\}E\left\{(x_j(t)-\overline{z})^{-1}\right\}\right). \end{aligned} \tag{3.48}$$

Then repeating the arguments (3.41)-(3.43), we obtain

$$\begin{aligned} &E\left\{\left|g_n(z,t) - E\{g_n(z,t)\}\right|^2\right\} \\ &\quad \le C(t)n^{-1} + n^{-2}\sum_{i,j=1} E\left\{\left(F_i(\tilde\sigma_n,z) - E\{F_i(\tilde\sigma_n,z)\}\right)\left(F_j(\tilde\sigma_n,z) - E\{F_j(\tilde\sigma_n,z)\}\right)\right\}, \end{aligned} \tag{3.49}$$

where $\tilde\sigma_n$ is defined by (3.36) and we denote

$$F_i(\sigma,z) = \frac{1}{\sqrt{2\pi}}\int\int\frac{d\nu_i(x)dye^{-y^2/2}}{x+c_i+\sigma^{1/2}y - z}.$$

Since evidently

$$|F_i(\sigma_1,z)-F_i(\sigma_2,z)|\le C\frac{|\sigma_1^{1/2}-\sigma_2^{1/2}|}{|\Im z|^2},$$

we get from (3.49)

$$E\left\{\left|g_n(z,t)-E\{g_n(z,t)\}\right|^2\right\}\le C(t)n^{-1}+C|\Im z|^{-2}E\{(\tilde\sigma_n-E\{\tilde\sigma_n\})^2\}. \tag{3.50}$$

Now the assertion of Lemma 4 follows from (3.38).

Using (3.35), the s.a. properties (3.33), and the fact that the system (2.15) is symmetric with respect to $x_1,\dots,x_{[fn]}$ and with respect $x_{[fn]+1},\dots,x_n$, we obtain

$$\begin{aligned}E\{\tilde\sigma_n(t)\}=\frac{\sigma_I[fn]}{n}E\left\{\left(\int_0^t dsx_1(s)\right)^2\right\}+\sigma_E(1-[fn]/n)E\left\{\left(\int_0^t dsx_n(s)\right)^2\right\}-\\ \sigma_*^{-1}\left(\frac{\sigma_I[fn]}{n}\int_0^t dsE\{x_1(s)\}+\sigma_E(1-[fn]/n)\int_0^t dsE\{x_1(s)\}\right)^2+o(1).\end{aligned} \tag{3.51}$$

Repeating our conclusions for $x_n(t)$, we get

$$x_n(t)=x_n(0)+\sum_{j=2}^{n}\frac{W_{nj}}{n^{1/2}}d_j^{(n)}(t)+\varepsilon_n^{(n)}(t), \tag{3.52}$$

where

$$E\{(\varepsilon_n^{(n)}(t))^2\}\le C(t)n^{-1}, \tag{3.53}$$

$$d_i^{(n)}(t)=\int_0^t x_i^{(n)}(s)ds\quad (i=1,\dots,n-1),$$

$x_i^{(n)}=(e^{t\mathbf{J}^{(n)}}\mathbf{x}(0))_i$ and the matrix $\mathbf{J}^{(n)}$ is obtained from $\mathbf{J}$ by replacing the last line and the last column by zeros. Then, applying Lemma 1, we obtain that the second sum in (3.52) is a Gaussian random variable with the same variance $\tilde\sigma_n(t)$ (see (3.51)).

Equations (3.31) and (3.52) combined with (2.16) give us

$$E\{x_1(t)\}=c_I,\quad E\{x_n(t)\}=c_E. \tag{3.54}$$

Denoting

$$R_n^{(1)}(t,s)=E\{x_1(t)x_1(s)\},\quad R_n^{(2)}(t,s)=E\{x_n(t)x_n(s)\}, \tag{3.55}$$

we obtain from (3.31) and (3.52) the system of equations

$$\begin{aligned}R_n^{(1)}(t,s) &= E\{x_1^2(0)\}+\sigma_If\int_0^t\int_0^s R_n^{(1)}(t',s')dt'ds'\\ &\quad+\sigma_E(1-f)\int_0^t\int_0^s R_n^{(2)}(t',s')dt'ds'-ts\sigma_*^{-1}(\sigma_Ifc_1+\sigma_E(1-f)c_2)^2+o(1),\\ R_n^{(2)}(t,s) &= E\{x_n^2(0)\}+\sigma_If\int_0^t\int_0^s R_n^{(1)}(t',s')dt'ds'\\ &\quad+\sigma_E(1-f)\int_0^t\int_0^s R_n^{(2)}(t',s')dt'ds'-ts\sigma_*^{-1}(\sigma_Ifc_1+\sigma_E(1-f)c_2)^2+o(1).\end{aligned} \tag{3.56}$$

Then we obtain that the function

$$R_n^{(0)}(t,s)=\sigma_IfR_n^{(1)}(t,s)+\sigma_E(1-f)R_n^{(2)}(t,s)-\sigma_*^{-1}(\sigma_Ifc_1+\sigma_E(1-f)c_2)^2, \tag{3.57}$$

satisfies the equation

$$R_n^{(0)}(t,s) = A + \sigma_* \int_0^t \int_0^s R_n^{(0)}(t',s')dt'ds' + o(1), \tag{3.58}$$

where

$$A = \sigma_I f E\{x_1^2(0)\} + \sigma_E(1-f)E\{x_n^2(0)\} - \sigma_*^{-1}(\sigma_I f c_1 + \sigma_E(1-f)c_2)^2 = \tag{3.59}$$
$$= \frac{\sigma_I\sigma_E f(1-f)(c_1-c_2)^2}{\sigma_*} + (\sigma_I\sigma_I^{(0)} f + \sigma_E\sigma_E^{(0)}(1-f)).$$

As it was proved in [7] this equation has the unique solution

$$R_n^{(0)}(t,s) = A\left(1 + \sum_{m=1}^{\infty} \frac{\sigma_*^m t^m s^m}{m!m!}\right) + o(1). \tag{3.60}$$

Then we can easily find that

$$\begin{aligned} R_n^{(1)}(t,s) &= E\{x_1^2(0)\} + A\sigma_*^{-1}\sum_{m=1}^{\infty}\frac{\sigma_*^m t^m s^m}{m!m!} + o(1), \\ R_n^{(2)}(t,s) &= E\{x_n^2(0)\} + A\sigma_*^{-1}\sum_{m=1}^{\infty}\frac{\sigma_*^m t^m s^m}{m!m!} + o(1). \end{aligned} \tag{3.61}$$

Hence

$$\tilde{\sigma}(t) = \lim_{n\to\infty}\int_0^t\int_0^t dt'ds' R_n^{(0)}(t',s') = A\sigma_*\sigma_*^{-1}\sum_{m=1}^{\infty}\frac{\sigma_*^m t^{2m}}{m!m!}. \tag{3.62}$$

Using (3.45), and the symmetry of the problem we obtain that $\mathcal{N}_n(t,\lambda)$ converges in probability to

$$fE\{\theta(\lambda - x_1(t))\} + (1-f)E\{\theta(\lambda - x_n(t))\}.$$

But by the above arguments

$$\begin{aligned} E\{\theta(\lambda - x_1(t))\} &\to \int_{-\infty}^{\lambda}\frac{dy}{\sqrt{2\pi}}\int d\nu_I(x)\exp\{(x - \tilde{\sigma}^{1/2}(t)y - c_I)^2/2\}, \\ E\{\theta(\lambda - x_n(t))\} &\to \int_{-\infty}^{\lambda}\frac{dy}{\sqrt{2\pi}}\int d\nu_E(x)\exp\{(x - \tilde{\sigma}^{1/2}(t)y - c_E)^2/2\}. \end{aligned}$$

Theorem 1 follows.

Proof of Theorem 2. To prove Theorem 2 it is convenient to consider the system (2.15) in the basis $\{\mathbf{u}_i\}_{i=1}^n$ defined above (see (3.7)-(3.9). Let

$$y_i(t) = (\mathbf{x}(t), \mathbf{u}_i) \tag{3.63}$$

Then the system (2.15) takes the form

$$\mathbf{y}' = \tilde{\mathbf{J}}\mathbf{y}, \tag{3.64}$$

where $\tilde{\mathbf{J}}$ is defined by (3.10). The question of interest is the behavior of $y_2(t)$. Repeating for $y_2(t)$ the arguments (3.19)-(3.24) we get the representation

$$y_2(t) = y_2(0) + \tilde{J}_{22}\int_0^t y_2(s)ds + \sum_{j=3}^{n}\frac{\tilde{W}_{1j}}{n^{1/2}}d_j(t) + \int_0^t ds\, r_n^{(2)}(t-s)y_2(s), \tag{3.65}$$

with

$$y_2(0) = (x(0), m) = \sqrt{n}(c_I\mu_I f + c_E\mu_E(1-f)) + \mu_I \sum_{i=1}^{[fn]} \frac{x_i^{(0)}}{\sqrt{n}} + \mu_E \sum_{i=[fn]+1}^{n} \frac{x_i^{(0)}}{\sqrt{n}}, \tag{3.66}$$

$$d_j(t) = \int_0^t ds\, y_j^{(2)}(s), \quad y_i^{(2)} = (e^{t\tilde{\mathbf{J}}^{(2)}}\mathbf{y}(0))_i,$$
$$r_n^{(2)}(t) = \int_0^t \tilde{r}_n^{(2)}(\tau)d\tau + n^{-1/2}W_{22}, \quad \tilde{r}_n^{(2)}(t) = n^{-1}\sum_{i,j=3}^{n}(e^{t\tilde{\mathbf{J}}^{(2)}})_{ij}\tilde{W}_{2i}\tilde{W}_{j2},$$

where and $\tilde{\mathbf{J}}^{(2)}$ is the matrix which we obtain from $\tilde{\mathbf{J}}$ replacing the first and the second lines and the first and the second columns by zeros. Taking the expectation in (3.65) we get the first statement of Theorem 2. Now assume that $c_I = c_E = c$. Then, repeating arguments (3.22)-(3.28), we get that

$$y_2(t) = y_2(0) + \sum_{j=3}^{n} \frac{\tilde{W}_{1j}}{n^{1/2}} d_j(t) + \tilde{\varepsilon}_n(t), \tag{3.67}$$

and

$$E\{(\tilde{\varepsilon}_n^{(2)}(t))^2\} \le C(t)n^{-1}. \tag{3.68}$$

Applying Central Limit Theorem to the r.h.s. of (3.66) it is easy to obtain that $y_2(0)$ converges in distribution to a Gaussian random variable with zero mean and the variance

$$\sigma_y = f\mu_I^2\sigma_I + (1-f)\mu_E^2\sigma_E = (1-f)\sigma_I + f\sigma_E.$$

Besides, since $\{\tilde{W}_{2,j}\}$ are independent Gaussian random variables, the sum in the r.h.s. of (3.65) is a gaussian random variable with the variance

$$\tilde{\sigma}^{(1)}(t) = n^{-1}\sigma_I \sum_{j=3}^{[fn]+1} (d_j^{(2)}(t))^2 + n^{-1}\sigma_E \sum_{j=[fn]+2}^{n} (d_j^{(2)}(t))^2 =$$
$$= n^{-1}\sigma_I \sum_{j=3}^{[fn]+1} \int_0^t\int_0^t dt'ds' y_j(t')y_j(s') + n^{-1}\sigma_E \sum_{j=[fn]+2}^{n} \int_0^t\int_0^t dt'ds' y_j(t')y_j(s') + o(1) = \tag{3.69}$$
$$= n^{-1}\sigma_I \int_0^t\int_0^t dt'ds' \sum_{k,k'=1}^{n} x_k(t')x_{k'}(s') \sum_{j=3}^{[fn]+1} u_{kj}u_{k'j} + n^{-1}\sigma_E \int_0^t\int_0^t dt'ds' \sum_{k,k'=1}^{n} x_k(t')x_{k'}(s') \sum_{j=[fn]+2}^{n} u_{kj}u_{k'j}.$$

If $1 \le k \le [fn]$, while $[fn]+1 \le k' \le n$ or $1 \le k' \le [fn]$, while $[fn]+1 \le k \le n$, then by construction of $\mathbf{u}_3, \dots, \mathbf{u}_n$

$$\sum_{j=3}^{[fn]+1} u_{kj}u_{k'j} = 0, \qquad \sum_{j=[fn]+2}^{n} u_{kj}u_{k'j} = 0.$$

Let $1 \le k, k' \le [fn]$. Then

$$\sum_{j=[fn]+2}^{n} u_{kj}u_{k'j} = 0$$

and so

$$\sum_{j=3}^{[fn]+1} u_{kj}u_{k'j} = \sum_{j=3}^{n} u_{kj}u_{k'j} = \delta_{k,k'} - u_{k1}u_{k'1} - u_{k2}u_{k'2} = \delta_{k,k'} - n^{-1}(1+\mu_I^2).$$

Similarly for $[fn]+1 \leq k, k' \leq n$, we get

$$\sum_{j=[fn]+2}^{n} u_{kj} u_{k'j} = \delta_{k,k'} - n^{-1}(1+\mu_E^2).$$

Finally we get

$$\tilde{\sigma}^{(1)}(t) = \tag{3.70}$$

$$= n^{-1}\sigma_I \int_0^t \int_0^t dt'ds' \sum_{k,k'=1}^{[fn]} x_k(t')x_{k'}(s')(\delta_{k,k'} - n^{-1}(1+\mu_I^2))+$$

$$+n^{-1}\sigma_E \int_0^t \int_0^t dt'ds' \sum_{k,k'=[fn]+1}^{n} x_k(t')x_{k'}(s')(\delta_{k,k'} - n^{-1}(1+\mu_E^2) =$$

$$= f\sigma_I \int_0^t \int_0^t dt'ds' R^{(1)}(t',s') + (1-f)\sigma_E \int_0^t \int_0^t dt'ds' R^{(2)}(t',s') - t^2(\sigma_I f^2 c^2(1+\mu_I^2) + \sigma_E(1-f)^2 c^2(1+\mu_E^2)) =$$

$$= \int_0^t \int_0^t dt'ds' R^{(0)}(t',s') = A\sigma_*^{-1} \sum_{m=1}^{\infty} +o(1). \frac{\sigma_*^m t^m s^m}{m!m!}$$

Since the sum of independent Gaussian variables is a Gaussian random variable with the variance equal to the sum of variances, we obtain that $w_n(t)$ converge in distribution to a Gaussian random variable with zero mean and the variance

$$\tilde{\sigma}^{(0)} = \tilde{\sigma}^{(1)}(t) + \sigma_y.$$

The second assertion of Theorem 2 follows.

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